Int. J. Heat Mass Transfer. Vol. 30, No. 1, pp. 199–201, 1987
Printed in Great Britain

0017–9310/87 $3.00+0.00
Pergamon Journals Ltd.

# TECHNICAL NOTES

## Determination of thermal parameters of relaxation materials

J. Gembarovič and V. Majerník
Katedra fyziky Pedagogickej fakulty, CS—949 74 Nitra, Czechoslovakia

(*Received* 14 *October* 1985 *and in final form* 10 *March* 1986)

It is well known that for a wide class of materials with memory (we will call them relaxation materials), the heat conduction can be adequately described only by the generalized heat conduction equation [1, 2]

$$\tau\frac{\partial^2 T}{\partial t^2}+\frac{\partial T}{\partial t}=a\Delta T \tag{1}$$

where $T = T(x, y, z, t)$ is the temperature in space–time point $x$, $y$, $z$, $t$; $a$ is the coefficient of the thermal diffusivity; and $\tau$ is so-called relaxation time. Equation (1) can be rearranged in the form

$$\frac{1}{\omega^2}\frac{\partial^2 T}{\partial t^2}+\frac{1}{a}\frac{\partial T}{\partial t}=\Delta T \tag{2}$$

which represents the dissipative wave equation with the speed of propagation of the thermal disturbance determined as

$$\omega=\sqrt{\frac{a}{\tau}}. \tag{3}$$

Since the equation of heat conduction (1) contains two functionally connected parameters $a$ and $\tau$, one has to modify the formulas needed for the experimental determination of the thermal diffusivity $a$ of the relaxation materials which generally differ from the known formulas derived from the standard heat conduction equation, i.e. the equation

$$\frac{\partial T}{\partial t}=a\Delta T. \tag{4}$$

The purpose of this note is to derive the formulas for the temperature field in samples of relaxation materials; this is necessary in order to determine experimentally the values of the thermal diffusivity and, simultaneously, the relaxation time by non-stationary experimental methods, especially by the well-known flash method [3].

In what follows we assume the standard experimental arrangement commonly used in the flash method of measurement of the thermal parameters. We consider a sample of relaxation material with thermal conductivity $k$ and relaxation time $\tau$. The sample has the form of a slab with thickness $L$. Let the initial temperature distribution be uniform, and the sample be adiabatically isolated from the ambient. The temperature rise $\theta(x, t)$ due to the absorption of an instantaneous pulse of energy $Q$ at the time $t = 0$ on the front face $x = 0$, can be found as a solution of (1) in the form

$$\tau\frac{\partial^2\theta}{\partial t^2}+\frac{\partial\theta}{\partial t}-a\frac{\partial^2\theta}{\partial x^2}=0 \quad 0\leqslant x\leqslant L \tag{5}$$

with the boundary conditions typical for the flash method

$$\theta(x,0)=0 \tag{6}$$

$$\left.\frac{\partial\theta}{\partial t}\right|_{t=0}=0 \tag{7}$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0}=-\frac{Q}{k}\delta(t) \tag{8}$$

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=L}=0. \tag{9}$$

If we apply the Laplace transformation to (5) by taking into account the initial conditions (6) and (7), we obtain the following subsidiary equation

$$a\frac{\mathrm{d}^2\bar{\theta}}{\mathrm{d}x^2}-p(p\tau+1)\bar{\theta}=0;\quad 0\leqslant x\leqslant L \tag{10}$$

with the conditions

$$\left.\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}x}\right|_{x=0}=-\frac{Q}{k}\quad\text{and}\quad\left.\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}x}\right|_{x=L}=0 \tag{11}$$

where

$$\bar{\theta}=\int_0^\infty \theta(x,t)\exp\{-pt\}\,\mathrm{d}t$$

is the Laplace transform of the temperature rise $\theta(x, t)$. The solution of (10) with respect to (11) is

$$\bar{\theta}(x,q)=\frac{Q}{k}[1-\exp(-2qL)]^{-1}\times\left\{\frac{\exp(-qx)}{q}+\frac{\exp[-q(2L-x)]}{q}\right\} \tag{12}$$

where

$$q=\sqrt{\frac{\tau}{a}}\sqrt{p\left(p+\frac{1}{\tau}\right)}. \tag{13}$$

Since for the flash method the solution of (5) is the most interesting for small values of time, we expand the RHS of (12) into a series

$$\bar{\theta}(x,q)=\frac{Q}{k}\sum_{m=0}^{\infty}\frac{1}{q}\left\{\exp[-q(2mL+x)]+\exp[-q(2mL+2L-x)]\right\}. \tag{14}$$

Using the Table of Transforms [4] we can find the inverse transformation of (14):

$$\begin{aligned}\theta(x,t)=\frac{Q}{k}\sqrt{\frac{a}{\tau}}\mathrm{e}^{-t/2\tau}\sum_{m=0}^{\infty}\Bigg\{&I_0\left(\frac{t}{2\tau}\sqrt{1-\frac{(2mL+x)^2\tau}{at^2}}\right)\\&\times\mu\left[t-\sqrt{\frac{\tau}{a}}(2mL+x)\right]\\&+I_0\left(\frac{t}{2\tau}\sqrt{1-\frac{(2mL+2L-x)^2\tau}{at^2}}\right)\\&\times\mu\left[t-\sqrt{\frac{\tau}{a}}(2mL+2L-x)\right]\Bigg\}\end{aligned} \tag{15}$$

## NOMENCLATURE

| | |
|---|---|
| $a$ | thermal diffusivity |
| $k$ | thermal conductivity |
| $L$ | thickness of the sample |
| $Q$ | amount of energy of the pulse |
| $t^*$ | time needed for appearance of the front of temperature disturbance in the plane $x = L$ |
| $T$ | temperature. |

Greek symbols

| | |
|---|---|
| $\tau$ | relaxation time |
| $\omega$ | speed of propagation of the thermal disturbance |
| $\theta(x, t)$ | temperature rise |
| $\bar{\theta}$ | Laplace transform of the temperature rise. |

where $I_0(z)$ is the modified Bessel function of the first kind, zero order, and $\mu(t-t')$ is unit step function:

$$\mu(t-t') = 1 \quad t > t'$$
$$= 0 \quad t < t'.$$

In the limiting case when $\tau \to 0$, (5) assumes the form of the solution of the standard heat conduction equation

$$\frac{\partial\theta}{\partial t} - a\frac{\partial^2\theta}{\partial x^2} = 0; \quad 0 \leqslant x \leqslant L$$

and the temperature rise distribution goes over the well-known formula [5]

$$\theta(x, t) = \frac{Q\sqrt{a}}{k\sqrt{\pi t}} \sum_{m=0}^{\infty} \left\{ \exp\left[ -\frac{(2mL+x)^2}{4at} \right] + \exp\left[ -\frac{(2mL+2L-x)^2}{4at} \right] \right\}.$$

The first term in (15) describes the flow of heat from the point $x = 0$ to the right, and the second one describes the waves reflected from the plane $x = L$ and moving to the left. The equilibrium temperature rise due to input of heat $Q$ J m$^{-2}$ is $Q/\rho cL$, where $\rho$ is the mass density, and $c$ is the specific heat capacity of the sample. If we want to express the temperature rise in the sample as a fraction of this equilibrium temperature, then using the relation between the $a$ and $k$ ($k = \rho ac$), we can rewrite (15) into dimensionless form

$$\theta = \frac{L}{\sqrt{a\tau}} e^{-t/2\tau} \sum_{m=0}^{\infty} \left\{ I_0\left( \frac{t}{2\tau}\sqrt{1 - \frac{(2mL+x)^2\tau}{at^2}} \right) \times \mu\left[ t - \sqrt{\frac{\tau}{a}}(2mL+x) \right] + I_0\left( \frac{t}{2\tau}\sqrt{1 - \frac{(2mL+2L-x)^2\tau}{at^2}} \right) \times \mu\left[ t - \sqrt{\frac{\tau}{a}}(2mL+2L-x) \right] \right\}. \quad (16)$$

From the experimental point of view the dependence of the temperature on time in the plane $x = L$ is of great interest. This dependence in our case can be written in the form

$$\theta(L, t) = \frac{2L}{\sqrt{a\tau}} e^{-t/2\tau} \sum_{m=0}^{\infty} I_0\left( \frac{t}{2\tau}\sqrt{1 - \frac{(2m+1)^2L^2\tau}{at^2}} \right) \times \mu\left[ t - \sqrt{\frac{\tau}{a}}(2m+1)L \right]. \quad (17)$$

Since in (17) there are two free parameters—thermal diffusivity $a$ and relaxation time $\tau$—for their simultaneous determination, it is advantageous to proceed as follows.

First, we determine experimentally the speed of propagation of the heat disturbance $\omega$ from the condition that $\theta(L,t)$ should be zero for $t < L/\omega$, and so $\omega$ is determined by the ratio $\omega = L/t^*$, where $t^*$ is the time needed for appearance of the front of temperature disturbance due to flash in the plane $x = L$. According to [3], equation (17) can be rewritten in the form

$$\theta(L, t) = \frac{2L\omega}{a} \exp\left( -\frac{\omega^2 t}{2a} \right) \sum_{m=0}^{\infty} I_0 \left( \frac{\omega^2 t}{2a}\sqrt{1 - \frac{(2m+1)^2L^2}{\omega^2 t^2}} \right) \mu\left[ t - \frac{(2m+1)L}{\omega} \right] \quad (18)$$

in which only the thermal diffusivity $a$ occurs. This can be determined from (18) if we know the experimental dependence of the temperature on the time in the plane $x = L$. Having the values of $a$ and $\omega$, we can then calculate the value of the relaxation time from equation (3).

This makes it possible to determine the value of thermal diffusivity as well as the relaxation time for relaxation materials by means of the flash method.

Let us make some remarks to the possibility of the measurement of the quantity $\omega$. According to Lykov [6] the typical value of $\tau$ for solids (metals) is of the order of $10^{-11}$ s. For example the speed of propagation of thermal disturbance of steel is equal to 1800 m s$^{-1}$. Generally the speed $\omega$ of the materials with phonon heat conductivity is approximately equal to their velocity of the sound. The value of $\omega$ for thermal insulating materials, e.g. porous or dispersion materials, is even less than that in the steel. Moreover, the speed $\omega$ of the materials in which phase transitions or chemical reactions take place, is further decreased due to the finiteness of the propagation of the front of phase transition or chemical reaction.

If we consider the thickness of the sample $L \sim 10^{-2}$ m then, under the plausible assumption that $\omega \sim 10^3$ m s$^{-1}$, the time $t^*$ is approximately equal to $10^{-5}$ s. Clearly it is practically impossible to register such a quick change of temperature with the usual experimental methods at room temperature. However, at sufficiently deep temperatures this thermal change can be reliably detected and registered with a relative precision of $10^{-3}$ by means of the sensors and detectors described in the works of Gutfeld and Nethercot [7]. This sensor consists of a thin metal layer which is at a temperature near to the transition to the superconducting state. By means of the measurement of the electrical resistance of this layer one can detect the time $t^*$ with high precision because the time constant of this sensor is of the order of $10^{-8}$ s [7]. We see that it is possible to measure the speed of propagation of thermal disturbance under the before-mentioned experimental conditions.

Using the new and finer sensors, which are under construction, it should also be possible to detect $\omega$ at considerably higher temperatures.

## REFERENCES

1. M. E. Gurtin, On the thermodynamics of materials with memory, *Archs ration. Mech. Analysis* **23**, 40 (1968).
2. J. W. Nunziato, On heat conduction in materials with memory, *Q. appl. Math.* **29**, 187 (1971).
3. W. J. Parker, R. J. Jenkins, C. P. Butler and G. L. Abbott, Flash method of determining thermal diffusivity, heat capacity and thermal conductivity, *J. appl. Phys.* **32**, 1679 (1961).

4. V. A. Ditkin and A. P. Prudnikov, *Spracovnik po operacionnomu izcisleniu* (in Russian), p. 265. Vysch. skola, Moskva (1965).
5. M. M. Klimenko, R. E. Krizanovski and V. E. Sherman, Impulznyj metod opredelenia Temperaturoprovodnosti, (in Russian), *Teplofiz. Vysok Temp.* **17,** 1216 (1979).
6. A. V. Lykov, *Teplomassoobmen* (in Russian), p. 451. Energia, Moskva (1978).
7. R. J. Gutfeld and A. H. Nethercot, *Phys. Rev. Lett.* **12,** 641 (1964); R. J. Gutfeld, *Physical Acoustics*, Vol. 5, p. 223. Academic Press, New York (1968).

---

*Int. J. Heat Mass Transfer.* Vol. 30, No. 1, pp. 201–203, 1987
Printed in Great Britain

0017-9310/87 $3.00+0.00
Pergamon Journals Ltd.

# Integral analysis of conjugate natural convection heat transfer from a long, vertical fin

K. HIMASEKHAR
Department of Mechanical Engineering and Applied Mechanics, University of Pennsylvania,
111A Towne Building/D3, Philadelphia, PA 19104, U.S.A.

(*Received* 21 *January* 1986 *and in final form* 21 *April* 1986)

## INTRODUCTION

BUOYANCY-induced convection is of great importance in many heat removal processes in technology. In particular, for low-power-level devices, it may be a significant cooling mechanism. In such cases, the transfer surface area may be increased, as in fins, for augmentation of heat transfer rates. In the thermal analysis of vertical fins, it is usually assumed that the fin is isothermal. This may be a reasonable assumption for short fins with high thermal conductance. However, long fins with low conductance would not be isothermal and for the estimation of heat transfer rates from such fins, the conjugate problem of conduction within the fin has to be solved simultaneously with natural convection in the ambient fluid. A numerical solution of this problem for a short plate fin in a fluid with $Pr = 0.72$ was obtained by Sparrow and Acharya [1]. Lock and Gunn [2] developed a similarity solution for a short, tapered fin in a fluid of infinite Prandtl number. Recently, Kuehn *et al.* [3] presented a similarity solution for the conjugate free convection heat transfer from a vertical fin of infinite length and obtained results for a uniform conductivity plate fin as a function of the fluid Prandtl number.

In the present work, an integral analysis has been carried out to obtain a closed-form solution for the heat transfer rates from a long, vertical fin with variable conductivity and/or thickness. The solution for the special case of fin with constant thickness and conductivity has been compared with that of Kuehn *et al.* [3] and a close agreement confirms the utility of the proposed equation.

## ANALYSIS AND RESULTS

Consider an infinitely long, vertical fin as shown in Fig. 1. The coordinate system used is also depicted in this figure. The base of the fin can be selected arbitrarily if the corresponding temperature $T_b$ is known [3]. The fin is at a higher temperature than that of the ambient fluid. The flow is assumed to be laminar. The Boussinesq approximation for the density variation is employed and the other fluid properties are taken to be constant. Further details of the problem can be found in ref. [3].

In this paper the case of the fin being hotter than the ambient fluid is explicitly considered. However, the analysis as well as the results also apply to the case of the fin being colder than the bulk fluid. In the latter case, the fin is to be inverted.

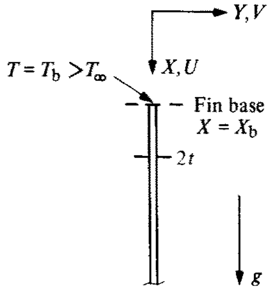

FIG. 1. The physical model.

The conservation equations for the fluid in the integral form are as follows [4]:

$$\frac{\mathrm{d}}{\mathrm{d}X}\int_0^\delta U^2\,\mathrm{d}Y = -\nu\left.\frac{\partial U}{\partial Y}\right|_{Y=0} - \int_0^\delta g\beta(T-T_\infty)\,\mathrm{d}Y \quad (1)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}X}\int_0^\delta U(T-T_\infty)\,\mathrm{d}Y = -\alpha\left.\frac{\partial T}{\partial Y}\right|_{Y=0}. \quad (2)$$

Energy balance for the fin with varying thickness and/or conductivity yields:

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(K_F t\frac{\mathrm{d}T_F}{\mathrm{d}X}\right) - h(T_F - T_\infty) = 0. \quad (3)$$

Here the thin fin approximation is employed. The boundary conditions for the fin are:

$$T_F = T_b \quad \text{at } X = X_b \quad (4a)$$

$$T_F = T_\infty \quad \text{as } X \to \infty. \quad (4b)$$

The temperature variation of the fin is taken as the power-law of the type [3]:

$$(T_F - T_\infty) = (T_b - T_\infty)(X/X_b)^n. \quad (5)$$

The following velocity and temperature distributions are assumed for the fluid [4]:

$$U/U_r = (Y/\delta)(1-Y/\delta)^2 \quad (6)$$

$$(T-T_\infty) = (T_F - T_\infty)(1-Y/\delta)^2. \quad (7)$$

Equations (1)–(3) are solved by using the following power-law variations for the boundary-layer thickness ($\delta$) and reference velocity ($U_r$):

$$\delta = c_1 x^{P1} \quad (8)$$

$$U_r = -c_2 x^{P2}. \quad (9)$$